(#) Izračunati pomoću diferenciranja po parametru integral

$$I(\alpha)=\int_0^{\frac{\pi}{2}} \ln(\sin^2 x+\alpha^2\cos^2 x)\,dx, \quad \alpha>0.$$

Rj. Ako je data f-ja dvije promjenjive $f(x,\alpha)$, ako su $f(x,\alpha)$ i $f'_\alpha(x,\alpha)$ neprekidne f-je tada za integral $I(\alpha)=\int_a^b f(x,\alpha)\,dx$ vrijedi $I'_\alpha(\alpha)=\int_a^b f'_\alpha(x,\alpha)\,dx$.

f'_α — predstavlja izvod f-je f po promjenjivoj α

$$I(\alpha)=\int_0^{\frac{\pi}{2}} \ln(\sin^2 x+\alpha^2\cos^2 x)\,dx$$

$$f(x,\alpha)=\ln(\sin^2 x+\alpha^2\cos^2 x)$$

$$f'_\alpha=\frac{1}{\sin^2 x+\alpha^2\cos^2 x}\cdot 2\alpha\cos^2 x=\frac{2\alpha\cos^2 x}{\sin^2 x+\alpha^2\cos^2 x}$$

$$I'_\alpha(\alpha)=\int_0^{\pi/2} f'_\alpha\,dx=\int_0^{\pi/2}\frac{2\alpha\cos^2 x}{\sin^2 x+\alpha^2\cos^2 x}\,dx \begin{matrix} /:\cos^2 x \\ /:\cos^2 x\end{matrix} =2\alpha\int_0^{\pi/2}\frac{dx}{tg^2 x+\alpha^2}$$

$$=\left|\begin{matrix} tg\, x=t & x=arctg\, t \\ x=0 \Rightarrow t=0 & dx=\dfrac{dt}{1+t^2} \\ x=\frac{\pi}{2} \Rightarrow t=\infty & \end{matrix}\right| = 2\alpha\int_0^{\infty}\frac{dt}{(t^2+\alpha^2)(t^2+1)}$$

$$\frac{1}{(x^2+\alpha^2)(x^2+1)}=\frac{Ax+B}{x^2+\alpha^2}+\frac{Cx+D}{x^2+1} \quad /(x^2+\alpha^2)(x^2+1)$$

$$1=A(x^3+x)+B(x^2+1)+C(x^3+\alpha^2 x)+D(x^2+\alpha^2)$$

$$\begin{aligned} A+C&=0 \quad (1)\\ B+D&=0 \quad (2)\\ A+\alpha^2 C&=0 \quad (3)\\ B+\alpha^2 D&=1 \quad (4)\end{aligned}$$

(1)-(4): $C-\alpha^2 C=0 \Rightarrow C=0 \Rightarrow A=0$

(2)-(4): $D-\alpha^2 D=-1 \qquad (\alpha^2-1)D=1$

$\alpha^2 D-D=1 \qquad D=\dfrac{1}{\alpha^2-1} \Rightarrow B=\dfrac{-1}{\alpha^2-1}$

$$I'_\alpha(\alpha) = 2\alpha \int_0^\infty \frac{dx}{(x^2+\alpha^2)(x^2+1)} = \frac{-2\alpha}{\alpha^2-1}\int_0^\infty \frac{dx}{x^2+\alpha^2} + \frac{2\alpha}{\alpha^2-1}\int_0^\infty \frac{dx}{x^2+1} =$$

$$- \frac{2\alpha}{\alpha^2-1} \cdot \frac{1}{\alpha} \operatorname{arctg} \frac{x}{\alpha}\Big|_0^\infty + \frac{2\alpha}{\alpha^2-1} \operatorname{arctg} x \Big|_0^\infty =$$

$$= - \frac{2}{\alpha^2-1}\left(\frac{\pi}{2} - 0\right) + \frac{2\alpha}{\alpha^2-1}\left(\frac{\pi}{2} - 0\right) =$$

$$= - \frac{\pi}{\alpha^2-1} + \frac{\pi}{\cancel{2}} \cdot \frac{\cancel{2}\alpha}{\alpha^2-1} = \frac{\pi(\alpha-1)}{\underbrace{\alpha^2-1}_{(\alpha-1)(\alpha+1)}} = \frac{\pi}{\alpha+1}$$

$$I'_\alpha(\alpha) = \frac{\pi}{\alpha+1} \Rightarrow I(\alpha) = \pi \ln|\alpha+1| + C = \Big| \text{kako je } \alpha>0 \Big|$$

$$= \pi \ln(\alpha+1) + C \quad \dots (*)$$

$$I(\alpha) = \int_0^{\pi/2} \ln(\sin^2 x + \alpha^2\cos^2 x)\, dx \Rightarrow I(1) = \int_0^{\pi/2} \ln(1)\, dx = 0 \quad \dots (**)$$

$$I(1) \overset{(*)}{=} \pi \ln 2 + C \overset{(**)}{=} 0 \Rightarrow C = -\pi \ln 2$$

$$\int_0^{\pi/2} \ln(\sin^2 x + \alpha^2\cos^2 x)\, dx = \pi \ln(\alpha+1) - \pi\ln 2 = \pi \ln \frac{\alpha+1}{2}$$

traženo rješenje

(#) Izračunati površinski integral

$$I = \iint_S \frac{dS}{(1+z)^2}$$

ako je S sfera $x^2+y^2+z^2=1$.

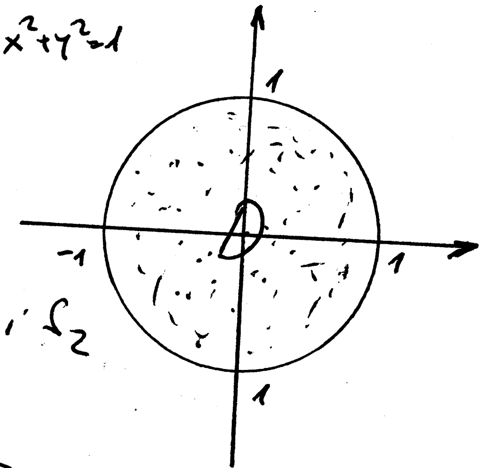
Rj. Zadatak se može uraditi na više načina

I način

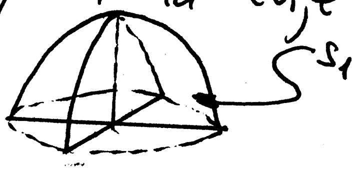
$z^2 = 1-x^2-y^2$

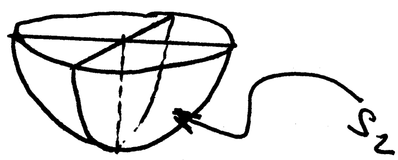
$z = \pm\sqrt{1-x^2-y^2}$

Presjek sfere $x^2+y^2+z^2=1$ sa xOy-ravni je krug $x^2+y^2=1$



U ovom slučaju sferu S ćemo podjeliti na dvije polusfere S_1 i S_2





$$I = \iint_S \frac{dS}{(1+z)^2} = \iint_{S_1} \frac{dS_1}{(1+z)^2} + \iint_{S_2} \frac{dS_2}{(1+z)^2}$$

gdje je $S_1: z = \sqrt{1-x^2-y^2}$

a $S_2 = -\sqrt{1-x^2-y^2}$

Znamo da je $dS_1 = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{1-x^2-y^2}}, \quad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{1-x^2-y^2}}, \quad dS_1 = \sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}\,dx\,dy$$

$$dS_1 = \sqrt{\frac{1}{1-x^2-y^2}} = \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy$$

Sad imamo

$$\iint\limits_{S_1} \frac{dS_1}{(1+z)^2} = \iint\limits_{D} \frac{1}{\left(1+\sqrt{1-x^2-y^2}\right)^2} \cdot \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy =$$

$$= \left| \begin{array}{l} \text{uvodimo polarne koordinate} \\ x=\rho\cos\varphi \\ y=\rho\sin\varphi \\ dx\,dy=\rho\,d\rho\,d\varphi \end{array} \quad D \xrightarrow{\text{transf.}} D_1 : \begin{cases} 0\le\rho\le 1 \\ 0\le\varphi\le 2\pi \end{cases} \quad x^2+y^2=\ldots=\rho^2 \right| = \iint\limits_{D_1} \frac{\rho\,d\rho\,d\varphi}{\left(1+\sqrt{1-\rho^2}\right)^2\sqrt{1-\rho^2}}$$

$$= \int_0^1 \frac{\rho\,d\rho}{\left(1+\sqrt{1-\rho^2}\right)^2\sqrt{1-\rho^2}} \int_0^{2\pi} d\varphi = \left| \begin{array}{l} 1-\rho^2=t^2 \\ -2\rho\,d\rho=2t\,dt \\ \rho\,d\rho=-t\,dt \end{array} \right| = 2\pi\int_1^0 \frac{-t\,dt}{(1+t)^2\cdot t}\,dt = \ldots = \pi$$

Slično

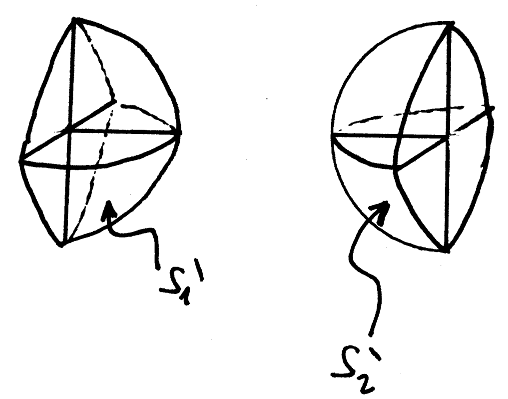
$$\iint\limits_{S_2} \frac{dS_2}{(1+z)^2} = \iint\limits_{D} \frac{1}{\left(1-\sqrt{1-x^2-y^2}\right)} \cdot \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = \left| \begin{array}{c} \text{uvodimo} \\ \text{polarne} \\ \text{koordinate} \end{array} \right| = \iint\limits_{D_1} \frac{\rho\,d\rho\,d\varphi}{\left(1-\sqrt{1-\rho^2}\right)^2\sqrt{1-\rho^2}}$$

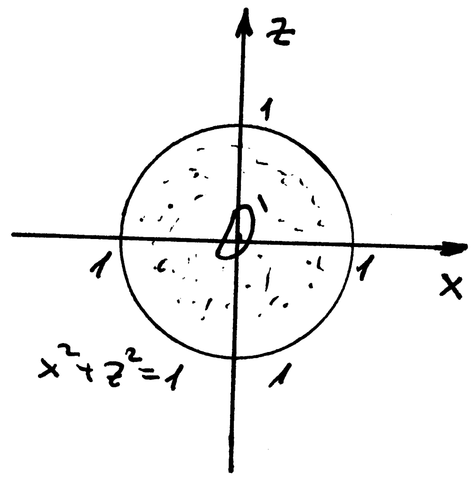
$$= 2\pi\int_0^1 \frac{\rho\,d\rho}{\left(1-\sqrt{1-\rho^2}\right)^2\sqrt{1-\rho^2}} = \left| \begin{array}{l} 1-\rho^2=t^2 \\ -2\rho\,d\rho=2t\,dt \\ \rho\,d\rho=-t\,dt \end{array} \right| = 2\pi\int_0^1 \frac{t\,dt}{(1-t)^2\,t} = \ldots = \infty$$

II način

$$y^2 = 1-x^2-z^2$$

$$y = \pm\sqrt{1-x^2-z^2}$$





$$I = \iint\limits_{S} \frac{dS}{(1+z)^2} = \iint\limits_{S_1'} \frac{dS_1'}{(1+z)^2} + \iint\limits_{S_2'} \frac{dS_2'}{(1+z)^2}$$

gdje je $S_1' : y=\sqrt{1-x^2-z^2}$

$S_2' : y=-\sqrt{1-x^2-z^2}$

$$y=\sqrt{1-x^2-z^2}, \quad \frac{\partial y}{\partial x} = \frac{-x}{\sqrt{1-x^2-z^2}}, \quad \frac{\partial y}{\partial z} = \frac{-z}{\sqrt{1-x^2-z^2}}, \quad 1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2 = \frac{1}{1-x^2-z^2}$$

$$\iint\limits_{S_1'} \frac{dS_1'}{(1+z)^2} = \iint\limits_{D} \frac{1}{(1+z)^2} \cdot \frac{dx\,dz}{\sqrt{1-x^2-z^2}} = \ldots$$ Slično i za S_2'

(#) Dokazati da je vektorsko polje

$$\vec{v} = (z\cos zx - y\sin x,\ \cos x,\ x\cos zx)$$

potencijalno i izračunati cirkulaciju tog polja duž prave od tačke $O(0,0,0)$ do tačke $A(1,2,\pi)$.

Rj.

Ako je $\mathrm{rot}\,\vec{v} = \vec{0}$ tada kažemo da je $\vec{v}$ potencijalno polje.

$$\mathrm{rot}\,\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\cos zx - y\sin x & \cos x & x\cos zx \end{vmatrix}$$

$$= \left(0-0,\ -(\cos zx - zx\cos zx - \cos zx + xz\cos zx),\ -\sin x + \sin x\right)$$

$$= (0,0,0) \Rightarrow \vec{v} \text{ je potencijalno polje}$$

$$C = \int_c \vec{v}\, d\vec{r} = \int_c V_x dx + V_y dy + V_z dz \quad \text{cirkulacija vektorskog polja}$$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$ jednačina prave kroz dvije tačke

$O(0,0,0)$
$A(1,2,\pi)$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{\pi} \quad (=t)$$

$$C = \int_{\overline{OA}} (z\cos zx - y\sin x)dx + \cos x\, dy + x\cos zx\, dz = \left| \overline{OA}: \begin{cases} x = t \\ y = 2t \\ z = \pi t \\ 0 \le t \le 1 \end{cases} \quad dx = dt,\ dy = 2dt,\ dz = \pi dt \right|$$

$$= \int_0^1 \left[\pi t\cos \pi t^2 - 2t\sin t + 2\cos t + \pi t\cos \pi t^2\right] dt$$

$$= \int_0^1 (2\cos t - 2t\sin t + 2\pi t\cos \pi t^2)\, dt = \ldots = 2\sin(1) + 2\cos(1) - 2\sin(1) + 0 = 2\cos 1$$